Journal of Engineering Mathematics **24**: 95–107, 1990.

On the free convection boundary layer on a vertical plate with prescribed surface heat flux

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Received 17 January 1989; accepted in revised form 4 September 1989

Abstract. The free convection boundary layer on a vertical plate with a prescribed surface heat flux proportional to $(1+x^2)^\mu$ (μ a constant) is discussed. For $\mu > -\frac{1}{2}$ the boundary-layer solution develops from a similarity solution valid for x small to the one valid for x large. However, with $\mu \leqslant -\frac{1}{2}$ the similarity equations for x large are not solvable and the behaviour for large x in this case is discussed. It is found that there are two cases to consider, namely $\mu < -\frac{1}{2}$ and $\mu = -\frac{1}{2}$. In both cases the leading-order problem is homogeneous involving an arbitrary constant which is determined from an integral property of the full boundary-layer problem. However, in the former case the asymptotic behaviour is algebraic, with the perturbation to the leading-order solution, arising from the heat flux boundary condition, being of $O[x^{1+2\mu}]$. The latter case also involves logarithmic terms, with the perturbation to be leading-order solution now being of $O[(\log x)^{-1}]$.

1. Introduction

In a recent paper [1] we considered the similarity solutions for free convection boundary-layer flow on a vertical plate with a prescribed surface heat flux. These similarity equations were derived originally by Sparrow and Gregg [2] and require a surface heating rate proportional to x^λ (where x is the co-ordinate measuring distance along the plate from the leading edge and λ is a constant). It was shown in [1] that the equations have a solution satisfying all the required boundary conditions only if $\lambda > -1$, with the solution becoming singular as $\lambda \to -1$; the nature of the singularity at $\lambda = -1$ was also discussed in [1].

In the context of more general free convection boundary-layer flows, these similarity solutions can be regarded as giving the behaviour near a leading edge (x small) or as asymptotic solutions (x large) with there being a transition between the two flow regions, obtained, say, by a numerical integration of the full boundary-layer problem. A solution procedure described, for example, by Merkin [3] or Hunt and Wilks [4]. A question that then arises is how would the boundary-layer solution develop if for small x, the solution were given by a similarity form for which a solution is possible (i.e. $\lambda > -1$) but attempted to reach asymptotic conditions for which a similarity solution were not possible (i.e. with $\lambda \leqslant -1$). It is this question that we attempt to answer in this paper.

To fix things we specify a (non-dimensional) prescribed wall heat flux

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = -(1+x^2)^\mu \tag{1}$$

where T and y are the (non-dimensional) temperature difference and normal co-ordinate respectively and μ is a constant. From (1) we see that, for $x \ll 1$, $(\partial T/\partial y)_{y=0} \simeq -1$, while, for $x \gg 1$, $(\partial T/\partial y)_{y=0} \simeq -x^{2\mu}$, so that though it is possible to write down similarity equations for both x small and x large, in the latter case these possess a solution only if we take $\mu > -\frac{1}{2}$.

As we shall see, for our discussion, the precise form that the surface heat flux takes is not important, only that it has the functional forms for x small and x large given above.

To start we obtain numerical solutions of the full boundary-layer problem with the prescribed wall heat flux given by (1) using the method of continuous transformations [4, 5]. These integrations show, as expected, that for values of $\mu > -\frac{1}{2}$, the solution develops smoothly from the similarity solution valid for x small to that appropriate for x large. However, when we take values of $\mu < -\frac{1}{2}$, we still find that the solution develops an asymptotic structure for x large, which cannot now be given by the corresponding similarity form. When we analyse this asymptotic behaviour in more detail we find that the leading-order solution is given by the similarity equations corresponding to taking $\mu = -\frac{1}{2}$, but now with homogeneous boundary conditions. This solution contains an arbitrary constant which is determined from an integral condition for the whole boundary-layer flow. The first perturbation to this leading-order problem arises from the application of the wall heat flux condition and is of $O(x^{1+2\mu})$.

This then leaves the final case which we need to consider, namely $\mu = -\frac{1}{2}$. Here we still find an asymptotic structure as for the case $\mu < -\frac{1}{2}$, but now this is modified by the inclusion of terms involving $\log x$, with, for example, the plate temperature being of $O(x^{-3/5}(\log x)^{4/5})$ for x large.

2. Equations

The (non-dimensional) equations governing the free convection boundary layer on a vertical plate are, from [1],

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y\,\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = T + \frac{\partial^3 \psi}{\partial y^3}\,, \tag{2a}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2}\,, \tag{2b}$$

where ψ is the stream function, defined in the usual way, and Pr is the Prandtl number. The boundary conditions to be applied are, from (1),

$$\begin{gathered}\psi = 0\,, \qquad \frac{\partial \psi}{\partial y} = 0\,, \qquad \frac{\partial T}{\partial y} = -(1+x^2)^{\mu} \quad \text{on } y = 0\,;\\ \frac{\partial \psi}{\partial y} \to 0\,, \qquad T \to 0 \quad \text{as } y \to \infty\,.\end{gathered} \tag{3}$$

To solve equations (2) numerically we follow Hunt and Wilks [4] or Kuiken [5] and make a composite transformation which reflects the similarity forms for both x small and x large and is an adaptation of the classical Görtler method to free convection boundary layers [6]. To do this we write

$$\psi = x^{\frac{4}{5}}(1+x^2)^{\frac{\mu}{5}}f(x,\eta)\,, \qquad T = x^{\frac{1}{5}}(1+x^2)^{\frac{4\mu}{5}}\theta(x,\eta)\,, \qquad \eta = y(1+x^2)^{\frac{\mu}{5}}/x^{\frac{1}{5}}\,. \tag{4}$$

Using (4), equations (2) become

$$\frac{\partial^3 f}{\partial\eta^3} + \theta + \frac{1}{1+x^2}\left[\left(\frac{4}{5} + \frac{2\mu}{5}\right)x^2 + \frac{4}{5}\right]f\frac{\partial^2 f}{\partial\eta^2} - \frac{1}{1+x^2}\left[\left(\frac{3}{5} + \frac{4\mu}{5}\right)x^2 + \frac{3}{5}\right]\left(\frac{\partial f}{\partial\eta}\right)^2 = x\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial x} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial\eta^2}\right), \tag{5a}$$

$$\frac{1}{\Pr}\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{1+x^2}\left[\left(\frac{4}{5} + \frac{2\mu}{5}\right)x^2 + \frac{4}{5}\right]f\frac{\partial\theta}{\partial\eta} - \frac{1}{1+x^2}\left[\left(\frac{1}{5} + \frac{8\mu}{5}\right)x^2 + \frac{1}{5}\right]\theta\frac{\partial f}{\partial\eta} = x\left(\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial\theta}{\partial\eta}\right), \tag{5b}$$

subject to the boundary conditions

$$f = \frac{\partial f}{\partial\eta} = 0, \quad \frac{\partial\theta}{\partial\eta} = -1 \quad \text{on } \eta = 0; \qquad \frac{\partial f}{\partial\eta} \to 0, \quad \theta \to 0 \quad \text{as } \eta \to \infty. \tag{6}$$

Writing $f = f(\eta)$, $\theta = \theta(\eta)$, we obtain, on putting $x = 0$ in equations (5),

$$f''' + \theta + \frac{4}{5}ff'' - \frac{3}{5}(f')^2 = 0, \tag{7a}$$

$$\frac{1}{\Pr}\theta'' + \frac{4}{5}f\theta' - \frac{1}{5}f'\theta = 0 \tag{7b}$$

(where primes denote differentiation with respect to η), which are the similarity equations corresponding to a uniform wall heat flux [2] while on letting $x \to \infty$, we get the equations

$$f''' + \theta + \frac{1}{5}(4 + 2\mu)ff'' - \frac{1}{5}(3 + 4\mu)(f')^2 = 0, \tag{8a}$$

$$\frac{1}{\Pr}\theta'' + \frac{2}{5}(2 + \mu)f\theta' - \frac{1}{5}(1 + 8\mu)f'\theta = 0, \tag{8b}$$

which correspond to a prescribed heat flux $(\partial T/\partial y)_{y=0} = -x^{2\mu}$, [1].

Equations (5) were solved numerically using essentially the same method used previously to solve other free convection problems, as described in some detail in [7] and the details need not be repeated fully here. The procedure is a marching method in which the derivations in the x-direction are replaced by forward differences and all the other terms averaged over the step from x to $x + \Delta x$. This results in two coupled ordinary differential equations which are then differenced using central differences and the resulting sets of nonlinear algebraic equations solved iteratively by the Newton–Raphson process. This was found to converge quickly, taking typically only 3 or 4 iterations to achieve convergence (the difference of any two successive iterates being everywhere less than 10^{-6}). A check was kept on the errors introduced into the scheme by differencing in the x-direction by going from x to $x + \Delta x$ in first one and then two steps and insisting that the difference between the two solutions was everywhere less than $5 \cdot 10^{-5}$. Otherwise the step length was halved and the

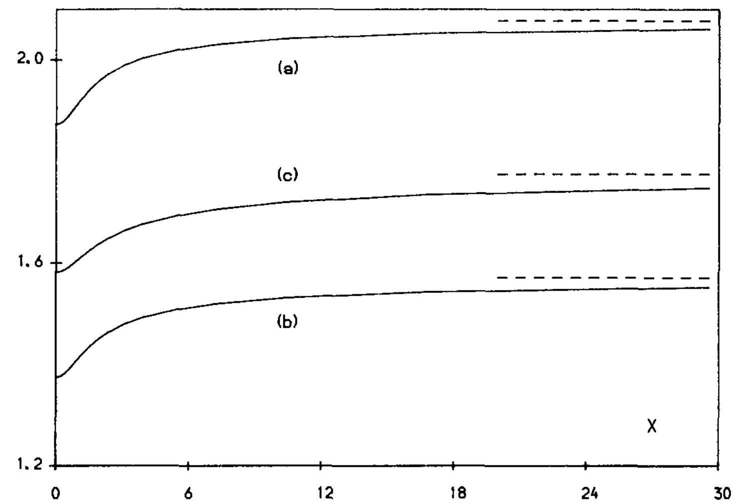

Fig. 1. Graphs of (a) $(\partial^2 f/\partial\eta^2)_{\eta=0}$, (b) $\theta(x, 0)$ and (c) $f(x, \infty)$, calculated from the numerical integration of equations (5) for $\mu = -\frac{1}{8}$.

process repeated until this accuracy criterion was achieved. The integrations started at $x = 0$ with the solutions given by equations (7) and proceeded stepwise for increasing x.

As a check we started with values of $\mu > -\frac{1}{2}$ and found, as expected, that the solution attained the asymptotic conditions as given by equations (8). This can be seen in Fig. 1 where we give graphs of $(\partial^2 f/\partial\eta^2)_{\eta=0}$, $\theta(x, 0)$ and $f(x, \infty)$ for the case $\mu = -\frac{1}{8}$ (corresponding to a uniform plate temperature for large x) and $\mathrm{Pr} = 1$ (throughout all the numerical results quoted are for $\mathrm{Pr} = 1$). We can see that all these quantities approach their asymptotic values (as given by the broken lines in Fig. 1) as x increases, $(\partial^2 f/\partial\eta^2)_{\eta=0}$, $\theta(x, 0)$ and $f(x, \infty)$ go from the values 1.3744, 1.8728 and 1.5820 at $x = 0$ to 1.5713, 2.0771 and 1.7757, respectively, for x large.

However, when we took values of $\mu \leq -\frac{1}{2}$, this was found not to be the case, with the numerical results not now settling onto some constant asymptotic values. It is the structure of the solution for x large with $\mu \leq -\frac{1}{2}$ that we discuss next. We find that there are two separate cases to consider, namely $\mu < -\frac{1}{2}$ and $\mu = -\frac{1}{2}$.

3. Asymptotic solution for $\mu < -\frac{1}{2}$

We first note that the transformation of variables which gives equations (8) is not appropriate for $\mu < -\frac{1}{2}$, as these equations do not then have a solution satisfying boundary conditions (6) [1]. An alternative is required and we now put

$$\psi = x^{\frac{3}{5}} F(x, \zeta), \qquad T = x^{-\frac{1}{5}} H(x, \zeta), \qquad \zeta = y/x^{\frac{2}{5}}. \tag{9}$$

Note that transformation (9) is the one that gives equations (8) for the critical case $\mu = -\frac{1}{2}$. Equations (2) become

$$\frac{\partial^3 F}{\partial\zeta^3} + H + \frac{3}{5} F \frac{\partial^2 F}{\partial\zeta^2} - \frac{1}{5}\left(\frac{\partial F}{\partial\zeta}\right)^2 = x\left(\frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial x\,\partial\zeta} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial\zeta^2}\right), \tag{10a}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 H}{\partial\zeta^2} + \frac{3}{5} F \frac{\partial H}{\partial\zeta} + \frac{3}{5} H \frac{\partial F}{\partial\zeta} = x\left(\frac{\partial F}{\partial\zeta}\frac{\partial H}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial H}{\partial\zeta}\right), \tag{10b}$$

with boundary conditions (3) becoming

$$F = \frac{\partial F}{\partial \zeta} = 0\,, \qquad \frac{\partial H}{\partial \zeta} = -x^{1+2\mu}\left(1 + \frac{1}{x^2}\right)^{\mu} \quad \text{on } \zeta = 0\,;$$
$$\frac{\partial F}{\partial \zeta} \to 0\,, \qquad H \to 0 \quad \text{as } \zeta \to \infty\,. \tag{11}$$

The form of the boundary conditions suggests looking for a solution of equations (10) by expanding in powers of $x^{1+2\mu}$ $(1 + 2\mu < 0)$. Hence we put

$$F(x, \zeta) = F_0(\zeta) + x^{1+2\mu}F_1(\zeta) + \cdots\,, \qquad H(x, \zeta) = H_0(\zeta) + x^{1+2\mu}H_1(\zeta) + \cdots\,.$$

When (12) is substituted into equations (10) we obtain at leading order

$$F_0''' + H_0 + \frac{3}{5}\,F_0F_0'' - \frac{1}{5}\,(F_0')^2 = 0\,, \tag{13a}$$

$$\frac{1}{\text{Pr}}\,H_0' + \frac{3}{5}\,F_0 H_0 = 0\,, \tag{13b}$$

subject to the homogeneous boundary conditions

$$F_0(0) = F_0'(0) = 0\,, \qquad H_0'(0) = 0\,;$$
$$F_0' \to 0\,, \qquad H_0 \to 0 \quad \text{as } \zeta \to \infty \tag{14}$$

(primes are now used to denote differentiation with respect to ζ). Equation (13b) has been obtained by integrating the equation derived from (10b) once, with boundary conditions (14) satisfied.

The homogeneous problem given by equations (13) and boundary conditions (14) has arisen previously [1, 8] with a solution $(\bar{F}_0, \bar{H}_0, \bar{\zeta})$ being obtained which has $\mathrm{d}^2\bar{F}_0/\mathrm{d}\bar{\zeta}^2 = 1$ on $\bar{\zeta} = 0$ (with $\bar{H}_0(0) = 0.73890$ and $\bar{F}_0(\infty) = 2.03449$ for $\text{Pr} = 1$). The general solution (F_0, H_0, ζ) which will have $\mathrm{d}^2F_0/\mathrm{d}\zeta^2 = k$ (say) on $\zeta = 0$ can then be found from this particular solution by the transformation

$$F_0 = k^{\frac{1}{3}}\bar{F}_0\,, \qquad H_0 = k^{\frac{4}{3}}\bar{H}_0\,, \qquad \zeta = k^{-\frac{1}{3}}\bar{\zeta}\,. \tag{15}$$

To fix the leading-order solution completely we need to determine the value of the constant k. This is achieved from an integral property of the full boundary-layer problem. To obtain this we integrate equation (2b) and apply boundary conditions (3) to get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^\infty \frac{\partial \psi}{\partial y}\,T\,\mathrm{d}y\right) = \frac{1}{\text{Pr}}\,(1 + x^2)^{\mu}\,. \tag{16}$$

Using the fact that the integral in (16) is zero at $x = 0$ (from transformation (4)), (16) then gives

$$\int_0^\infty \frac{\partial \psi}{\partial y}\,T\,\mathrm{d}y = \frac{1}{\text{Pr}} \int_0^x (1 + s^2)^{\mu}\,\mathrm{d}s\,. \tag{17}$$

For $\mu < -\frac{1}{2}$, it follows that for $x \gg 1$,

$$\int_0^x (1+s^2)^\mu \, ds = I_\infty + O(x^{2\mu+1}) , \tag{18}$$

where $I_\infty = \int_0^\infty (1+x^2)^\mu \, dx$. This infinite integral can be expressed in terms of the Beta function [9] as

$$I_\infty = \frac{1}{2} B\left(-\mu - \frac{3}{2}, -\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(-\mu - \frac{1}{2}\right)}{2\Gamma(-\mu)} . \tag{19}$$

Hence we have that

$$\int_0^\infty \frac{\partial \psi}{\partial y} T \, dy = I_\infty + O(x^{2\mu+1}) \tag{20}$$

for x large. Substituting transformation (9) in (20) results in

$$\int_0^\infty H \frac{\partial F}{\partial \zeta} \, d\zeta = \frac{1}{\Pr} \int_0^x (1+s^2)^\mu \, ds . \tag{21}$$

So that for the leading-order solution in expansion (12), (21) gives, using (15),

$$k^{\frac{5}{3}} \int_0^\infty \bar{F}_0' \bar{H}_0 \, d\bar{\zeta} = I_\infty . \tag{22}$$

Now the integral in expression (22) can be determined from our basic solution $(\bar{F}_0, \bar{H}_0, \bar{\zeta})$, and, for $\Pr = 1$, we find that $\int_0^\infty \bar{F}_0' \bar{H}_0 \, d\bar{\zeta} = 0.866367$, which in turn determines k as

$$k = 1.01369 \left(\frac{\Gamma(-\mu - \frac{1}{2})}{\Gamma(-\mu)} \right)^{\frac{3}{5}} . \tag{23}$$

With k given by (23), the leading-order solution (F_0, H_0, ζ) can then be found using (15).

We are now in a position to consider the next terms in expansion (12). The equations for the terms of $O(x^{1+2\mu})$ are

$$F_1''' + H_1 + \frac{3}{5} F_0 F_1'' - \left(\frac{7}{5} + 2\mu\right) F_0' F_1' + \left(\frac{8}{5} + 2\mu\right) F_0'' F_1 = 0 , \tag{24a}$$

$$\frac{1}{\Pr} H_1'' + \frac{3}{5} F_0 H_1' + \left(\frac{8}{5} + 2\mu\right) F_1 H_0' + \frac{3}{5} F_1' H_0 - \left(\frac{2}{5} + 2\mu\right) F_0' H_1 = 0 . \tag{24b}$$

Subject to the boundary conditions

$$\begin{aligned} &F_1(0) = F_1'(0) = 0 , \qquad H_1'(0) = -1 ; \\ &F_1' \to 0 , \qquad H_1 \to 0 \quad \text{as } \zeta \to \infty . \end{aligned} \tag{25}$$

Then, for a given μ, equations (24) can be solved for F_1 and H_1 with the solution then proceeding to higher-order terms. There is a modification to this expansion provided by the eigensolutions (F_e, H_e) at $O(x^{-1})$. These are, in effect, solutions of equations (24) (taking $\mu = -1$) which satisfy homogeneous boundary conditions and arise because of the arbitrariness in the location of the leading edge in the asymptotic expansion [10]. We find that

$$F_e = \alpha(2\zeta F_0' - 3F_0)\,, \qquad H_e = \alpha(2\zeta H_0' + 3H_0) \tag{26}$$

for any constant multiple α. The constant α is not determined from the integral condition (21), which gives, for this term (since there is, in general, no term of $O(x^{-1})$ in the asymptotic expansion of the integral $\int_0^x (1+s^2)^\mu\,ds$) that $\int_0^\infty (F_0'H_e + F_e'H_0)\,d\zeta = 0$. Substituting (26) into this integral (and integrating by parts) shows that this condition is satisfied identically with α then left arbitrary.

There is an exception to this when $\mu = -1$. In which case the eigensolutions and the perturbation to the leading order from the application of the wall heat flux boundary condition are both of the same order, $O(x^{-1})$. The existence of these eigensolutions then requires the expansion of $F(x, \zeta)$ and $H(x, \zeta)$ to take the form [11]

$$\begin{aligned} F(x,\zeta) &= F_0(\zeta) + \frac{\log x}{x} F_e(\zeta) + \frac{1}{x} F_1(\zeta) + \cdots\,, \\ H(x,\zeta) &= H_0(\zeta) + \frac{\log x}{x} H_e(\zeta) + \frac{1}{x} H_1(\zeta) + \cdots\,. \end{aligned} \tag{27}$$

This modification gives for the equations at $O(x^{-1})$,

$$F_1''' + H_1 + \frac{3}{5} F_0 F_1'' + \frac{3}{5} F_0' F_1' - \frac{2}{5} F_0'' F_1 = \alpha(3F_0 F_0'' - (F_0')^2)\,, \tag{28a}$$

$$\frac{1}{\mathrm{Pr}} H_1'' + \frac{3}{5} F_0 H_1' - \frac{2}{5} F_1 H_0' + \frac{3}{5} F_1' H_0 + \frac{8}{5} F_0' H_1 = 3\alpha(H_0' F_0 + F_0' H_0)\,. \tag{28b}$$

To solve equations (28) numerically we construct two complementary functions (F_a, H_a) and (F_b, H_b) which have $F_a''(0) = 1$, $H_a(0) = 0$, $H_a'(0) = 0$ and $F_b''(0) = 0$, $H_b(0) = 1$ and $H_b'(0) = 0$. Then we construct two particular integrals, the first (F_c, H_c) has $F_c''(0) = 0$, $H_c(0) = 0$, $H_c'(0) = -1$ and satisfies equations (28) with $\alpha = 0$, the second (F_d, H_d) has $F_d''(0) = 0$, $H_d(0) = 0$, $H_d'(0) = 0$ and satisfies equations (28) with $\alpha = 1$. The complete solution is then given by

$$\begin{aligned} F_1 &= aF_a + bF_b + F_c + \alpha F_d\,, \\ H_1 &= aH_a + bH_b + H_c + \alpha H_d\,, \end{aligned} \tag{29}$$

for constants a and b. Now, as $\zeta \to \infty$,

$$H_i \to A_i\,, \qquad F_i' \sim -\frac{5A_i}{3F_0(\infty)}\,\zeta + B_i \quad (i = a, b, c, d) \tag{30}$$

for constants A_i and B_i. To satisfy the boundary conditions on F_1 and H_1 as $\zeta \to \infty$ then requires

$$\begin{aligned} aA_a + bA_b + A_c + \alpha A_d &= 0\,, \\ aB_a + bB_b + B_c + \alpha B_d &= 0\,. \end{aligned} \tag{31}$$

However, the existence of eigensolutions (26) means that

$$A_a B_b - A_b B_b = 0 \tag{32}$$

which, when used in (31), gives an equation for α as

$$\alpha = \frac{B_c A_a - B_a A_c}{A_d B_a - B_d A_a} \, . \tag{33}$$

On performing the numerical integrations, we obtained, for $\mathrm{Pr} = 1$, $\alpha = 0.10623$. The solution at $O(x^{-1})$ is not unique as arbitrary multiples of the eigensolutions (F_e, H_e) can still be added in.

To complete the discussion of this case we compared the values of the (non-dimensional) skin friction $\tau_w = (\partial^2\psi/\partial y^2)_{y=0}$, plate temperature $\theta_w = T(x,0)$ and $\psi_\infty = \psi(x,\infty)$ as calculated from the numerical solution of equations (5) with the results obtained from the asymptotic series for the case $\mu = -1$. Graphs of these quantities are shown (plotted against log x) in Figs. 2. From (9) using the solution for F_0 and H_0 and the value of k found previously (for $\mu = -1$, $I_\infty = \frac{1}{2}\pi$) we have that

$$\tau_w \sim 1.4291 x^{-\frac{1}{5}} + \cdots, \qquad \theta_w \sim 1.894 x^{-\frac{3}{5}} + \cdots, \qquad \psi_\infty \sim 2.2916 x^{-\frac{3}{5}} + \cdots \tag{34}$$

for $x \gg 1$. Graphs of the asymptotic expressions given by (34) are also shown in Figs. 2 (by the broken lines) where we can see that they are all in very good agreement with the values obtained from the numerical solution, which acts as a good confirmation of the above theory. The $x^{-3/5}$ dependence of the plate temperature for x large also arises in wall plumes on adiabatic walls [12], and this can be thought of as the limiting form of our solution since $(\partial T/\partial y)_{y=0} \to 0$ as $x \to \infty$ for all $\mu < 0$. However, we find that this power-law variation of plate temperature for x large arises only for the case when $\mu < -\frac{1}{2}$. We now go on to complete the discussion by considering the case $\mu = -\frac{1}{2}$.

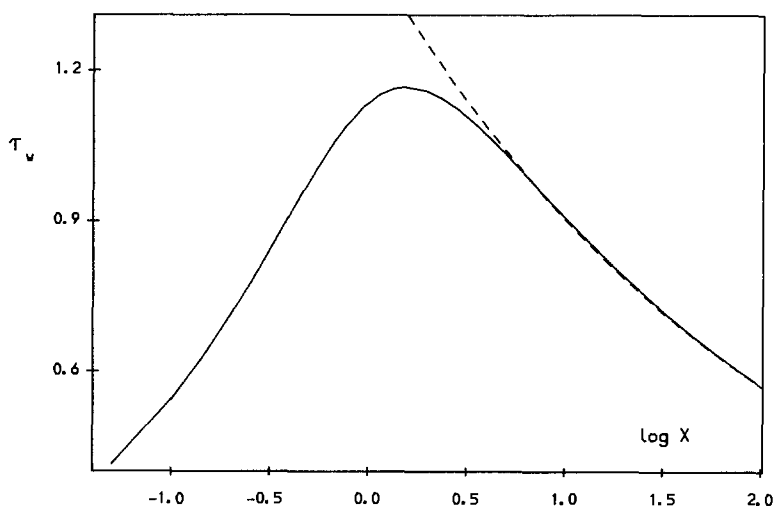

Fig. 2(a). Graphs of τ_w calculated from the numerical solution of equations (5) (full line) and from (34) (broken line) for $\mu = -1$.

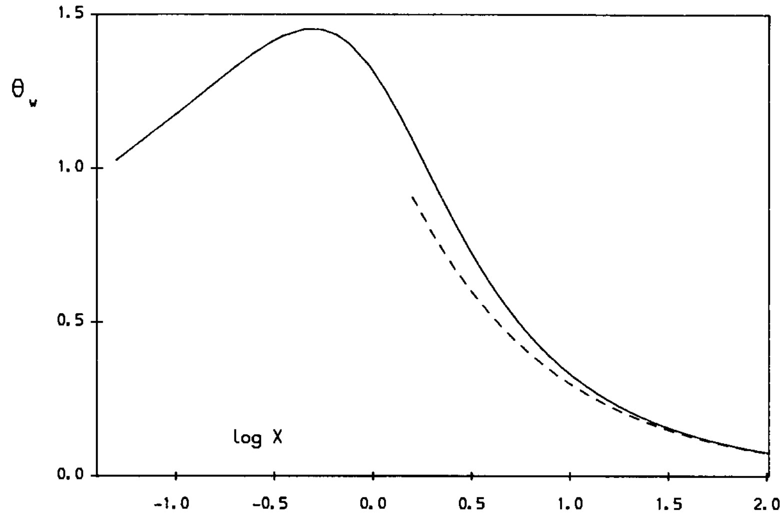

Fig. 2(b). Graphs of θ_w calculated from the numerical solution of equations (5) (full line) and from (34) (broken line) for $\mu = -1$.

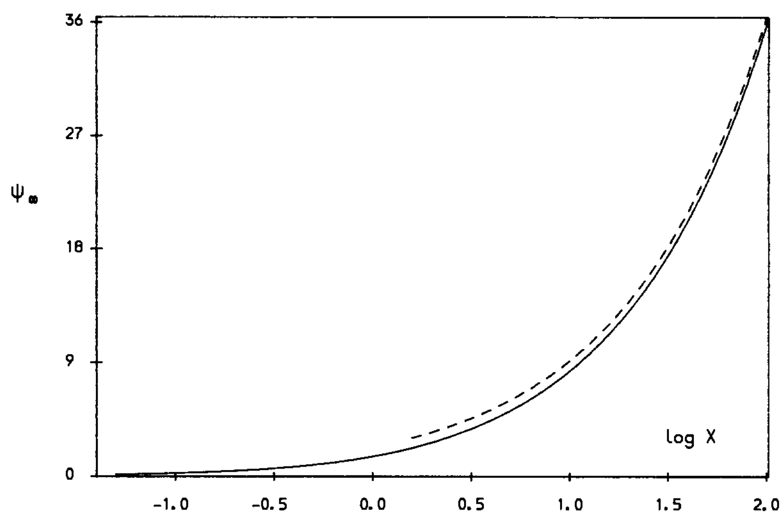

Fig. 2(c). Graphs of ψ_∞ calculated from the numerical solution of equations (5) (full line) and from (34) (broken line) for $\mu = -1$.

4. Asymptotic expansion for $\mu = -\frac{1}{2}$

Clearly the expansion in powers of $x^{1+2\mu}$ for $\mu < -\frac{1}{2}$ breaks down when $\mu = -\frac{1}{2}$ and an alternative approach is required. To get an insight into how we might proceed in this case we use integral condition (17), which becomes for $\mu = -\frac{1}{2}$,

$$\int_0^\infty \frac{\partial \psi}{\partial y} T \, \mathrm{d}y = \frac{1}{\mathrm{Pr}} \log(x + \sqrt{x^2 + 1}) \,. \tag{35}$$

Since $\log(x + \sqrt{x^2 + 1}) = \log x + \log 2 + O(x^{-2})$ as $x \to \infty$ this suggests that transformation

(9) should be modified to include terms in $\log x$. So we try

$$\psi = x^{\frac{3}{5}}(\log x)^{\alpha}\phi(x,\tau)\,, \qquad T = x^{-\frac{3}{5}}(\log x)^{\beta}h(x,\tau)\,, \qquad \tau = yx^{-\frac{2}{5}}(\log x)^{\gamma}\,. \tag{36a}$$

Result (35) gives $\alpha + \beta = 1$ and a balancing of terms in equations (2a) requires $\alpha + 3\gamma = \beta = 2\alpha + 2\gamma$. Hence $\alpha = \gamma = \frac{1}{5}$ and $\beta = \frac{4}{5}$ with (36a) becoming

$$\psi = x^{\frac{3}{5}}(\log x)^{\frac{1}{5}}\phi(x,\tau)\,, \qquad T = x^{-\frac{3}{5}}(\log x)^{\frac{4}{5}}h(x,\tau)\,, \qquad \tau = yx^{-\frac{2}{5}}(\log x)^{\frac{1}{5}}\,. \tag{36b}$$

Using (36b), equations (2) become

$$\frac{\partial^3\phi}{\partial\tau^3} + h + \left(\frac{3}{5} + \frac{1}{5\log x}\right)\phi\,\frac{\partial^2\phi}{\partial\tau^2} - \left(\frac{1}{5} + \frac{2}{5\log x}\right)\left(\frac{\partial\phi}{\partial\tau}\right)^2 = x\left(\frac{\partial\phi}{\partial\tau}\,\frac{\partial^2\phi}{\partial\tau\,\partial x} - \frac{\partial\phi}{\partial x}\,\frac{\partial^2\phi}{\partial\tau^2}\right), \tag{37a}$$

$$\frac{1}{\Pr}\,\frac{\partial^2 h}{\partial\tau^2} + \left(\frac{3}{5} + \frac{1}{5\log x}\right)\phi\,\frac{\partial h}{\partial\tau} + \left(\frac{3}{5} - \frac{4}{5\log x}\right)h\,\frac{\partial\phi}{\partial\tau} = x\left(\frac{\partial\phi}{\partial\tau}\,\frac{\partial h}{\partial x} - \frac{\partial\phi}{\partial x}\,\frac{\partial h}{\partial\tau}\right), \tag{37b}$$

with boundary conditions (3) becoming

$$\begin{aligned} &\phi = 0\,, \qquad \frac{\partial\phi}{\partial\tau} = 0\,, \qquad \frac{\partial h}{\partial\tau} = -\frac{1}{\log x}\left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}} \quad \text{on } \tau = 0\,; \\ &\frac{\partial\phi}{\partial\tau} \to 0\,, \qquad h \to 0 \quad \text{as } \tau \to \infty\,. \end{aligned} \tag{38}$$

Equations (37) and boundary conditions (38) suggest looking for a solution of equations (37) by expanding

$$\begin{aligned} \phi(x,\tau) &= \phi_0(\tau) + (\log x)^{-1}\phi_1(\tau) + \cdots\,, \\ h(x,\tau) &= h_0(\tau) + (\log x)^{-1}h_1(\tau) + \cdots\,. \end{aligned} \tag{39}$$

At leading order we have

$$\phi_0''' + h_0 + \frac{3}{5}\,\phi_0\phi_0'' - \frac{1}{5}\,(\phi_0')^2 = 0\,, \tag{40a}$$

$$\frac{1}{\Pr}\,h_0' + \frac{3}{5}\,\phi_0 h_0 = 0\,, \tag{40b}$$

satisfying the boundary conditions

$$\phi_0(0) = \phi_0'(0) = 0\,, \qquad h_0'(0) = 0\,; \qquad \phi_0' \to 0\,, \qquad h_0 \to 0 \quad \text{as } \tau \to \infty\,, \tag{41}$$

(primes are now used to denote differentiation with respect to τ and equation (40b) has been integrated once).

This problem is the same as (13) discussed previously and the general solution (ϕ_0, h_0, τ) can be obtained from the particular solution $(\bar{F}_0, \bar{H}_0, \bar{\zeta})$ by the transformation

$$\phi_0 = C^{\frac{1}{3}}\bar{F}_0\,, \qquad h_0 = C^{\frac{4}{3}}\bar{H}_0\,, \qquad \tau = C^{-\frac{1}{3}}\bar{\zeta}\,, \tag{42}$$

where we are taking $(d^2\phi_0/d\tau^2)_{\tau=0} = C$ for some constant C to be determined. Recall that the solution $(\bar{F}_0, \bar{H}_0, \bar{\zeta})$ was such that $(d^2\bar{F}_0/d\bar{\zeta}^2)_{\bar{\zeta}=0} = 1$.

To find C we use the transformation (36b) in the integral condition (35) to obtain

$$\int_0^\infty \phi h \, d\tau = \frac{1}{\Pr \log x} \log(x + \sqrt{x^2+1}) = \frac{1}{\Pr}\left(1 + \frac{\log 2}{\log x} + O\left(\frac{x^{-2}}{\log x}\right)\right). \tag{43}$$

So that, for the leading-order terms in expansion (39), we require

$$\int_0^\infty \phi_0' h_0 \, d\tau = 1 \quad \text{or} \quad C = \left(\int_0^\infty \bar{F}_0' \bar{H}_0 \, d\bar{\zeta}\right)^{-\frac{3}{5}}. \tag{44}$$

Using the previously calculated value for the integral in (44) gives

$$C = 1.08988. \tag{45}$$

The solution can then proceed to the higher-order terms in expansion (39). At $O((\log x)^{-1})$ we have the equations

$$\phi_1''' + h_1 + \frac{3}{5}\phi_0\phi_1'' - \frac{2}{5}\phi_0'\phi_1' + \frac{3}{5}\phi_0''\phi_1 = \frac{2}{5}(\phi_0')^2 - \frac{1}{5}\phi_0\phi_0'', \tag{46a}$$

$$\frac{1}{\Pr} h_1'' + \frac{3}{5}(\phi_0 h_1' + \phi_1 h_0' + \phi_0' h_1 + \phi_1' h_0) = \frac{4}{5}\phi_0' h_0 - \frac{1}{5}\phi_0 h_0', \tag{46b}$$

with boundary conditions

$$\phi_1(0) = \phi_1'(0) = 0, \qquad h_1'(0) = -1; \qquad \phi_1' \to 0, \qquad h_1 \to 0 \quad \text{as } \tau \to \infty. \tag{47}$$

To solve equations (46) numerically we construct a complementary function, (ϕ_a, h_a), which has $\phi_a''(0) = 1$ and $h_a(0) = h_a'(0) = 0$ and satisfies equations (46) with the right-hand sides put to zero. Since the homogeneous equation derived from (46b) can be integrated once to $h_a' + \frac{3}{5}(h_a\phi_0 + h_0\phi_a) = 0$ (satisfying all the imposed boundary conditions), it follows that for this complementary function, $h_a \to 0$, $\phi_a' \to D_a$ as $\tau \to \infty$ for some constant D_a. To this complementary function we add a particular integral, (ϕ_b, h_b), which satisfies equations (46) fully and has $\phi_b''(0) = 0$, $h_b(0) = 0$ and $h_b'(0) = -1$. Again we can integrate equation (46b) to get on applying the imposed boundary conditions

$$h_b' + 1 + \frac{3}{5}(\phi_0 h_b + \phi_b h_0) = \int_0^\tau \phi_0' h_0 \, d\bar{\tau} - \frac{1}{5}\phi_0 h_0. \tag{48}$$

Now, from (44), as $\tau \to \infty$, $\int_0^\tau \phi_0' h_0 \, d\bar{\tau} \to 1$, so that (48) becomes approximately, for τ large,

$$h_b' + \phi_0 h_b = 0 \tag{49}$$

where the terms neglected in (49) are all exponentially small. So that $h_b \to 0$, $\phi_b' \to D_b$ as $\tau \to \infty$ (for some further constant D_b). The numerical integrations give $D_a = 8.80458$ and $D_b = 7.73127$. The complete solution is then given by

$$\phi_1 = \gamma\phi_a + \phi_b, \qquad h_1 = \gamma h_a + h_b \tag{50}$$

where we need to choose the constant $\gamma = -D_b/D_a = -0.87810$ so as to satisfy the outer boundary conditions.

The solution at this stage is still not unique, as we can add in arbitrary multiples of the eigensolutions (ϕ_e, h_e), given by

$$\phi_e = \tau\phi_0' + \phi_0\,, \qquad h_e = \tau h_0' + 4h_0\,. \tag{51}$$

This multiple can then be determined using the integral condition (43) that $\int_0^\infty (h_0\phi_1' + h_1\phi_0')\,d\tau = \mathrm{Pr}^{-1}\log 2$.

The non-dimensional quantities τ_w, θ_w and ψ_∞ defined in the previous section can be calculated by using transformation (36b) and the value of C given by (44) as

$$\tau_w \sim 1.0899\,\frac{(\log x)^{\frac{3}{5}}}{x^{\frac{1}{5}}} + \cdots\,, \qquad \theta_w \sim 0.8288\,\frac{(\log x)^{\frac{4}{5}}}{x^{\frac{3}{5}}} + \cdots\,,$$
$$\psi_\infty \sim 2.0937(\log x)^{\frac{1}{5}}x^{\frac{3}{5}} + \cdots\,. \tag{52}$$

as $x \to \infty$. As a check on the above theory, we solved equations (5) numerically for the case $\mu = -\frac{1}{2}$, allowing the solution to proceed to very large values of x. We found, for this case, that the numerical solution did not quite settle onto an asymptotic limit as it did for the values of $\mu > -\frac{1}{2}$ but changed only very slowly as x increased. From this solution we calculated $\tau_w^* = x^{\frac{1}{5}}(\log x)^{-\frac{3}{5}}\tau_w$, $\theta_w^* = x^{\frac{3}{5}}(\log x)^{-\frac{4}{5}}\theta_w$ and $\psi_\infty^* = x^{-\frac{3}{5}}(\log x)^{-\frac{1}{5}}\psi_\infty$, the results are shown in Table 1. These results do appear to be approaching their respective asymptotic limits as given by (52) as x is increased, though the approach is rather slow. This is to be expected as the perturbation to the leading-order solution, which is all we have considered in (52), is of $O((\log x)^{-1})$ and at the final value of x given Table 1, $(\log x)^{-1}$ is only 0.0565, which is comparable with the difference between the values of τ_w^*, θ_w^* and ψ_∞^* given at this x and the corresponding asymptotic limits.

Table 1. Values of $\tau_w^* = x^{\frac{1}{5}}(\log x)^{-\frac{3}{5}}\tau_w$, $\theta_w^* = x^{\frac{3}{5}}(\log x)^{-\frac{4}{5}}\theta_w$ and $\psi_\infty^* = x^{-\frac{3}{5}}(\log x)^{-\frac{1}{5}}\psi_\infty$ as calculated from the numerical solution of equations (5)

x	τ_w^*	θ_w^*	ψ_∞^*
178	1.255	1.171	1.994
306	1.240	1.139	2.004
562	1.226	1.109	2.013
1586	1.208	1.070	2.025
3634	1.196	1.046	2.032
7730	1.187	1.028	2.037
24114	1.177	1.006	2.044
56882	1.170	0.992	2.048
122418	1.165	0.981	2.051
384562	1.158	0.968	2.055
908850	1.154	0.959	2.058
1957426	1.151	0.052	2.060
6151730	1.146	0.948	2.062
14540338	1.143	0.937	2.064
48094770	1.140	0.930	2.066
∞	1.090	0.829	2.094

References

1. J.H. Merkin, A note on the similarity solutions for free convection on a vertical plate, *J. Engng. Math.* 19 (1985) 189–201.
2. E.M. Sparrow and J.L. Gregg, Similar solutions for free convection from a nonisothermal vertical plate, *Trans. ASME* 80 (1958) 379–386.
3. J.H. Merkin, The effect of buoyancy forces on the boundary-layer flow over a semi-infinite vertical flat plate in a uniform stream, *J. Fluid Mech.* 35 (1969) 439–450.
4. R. Hunt and G. Wilks, Continuous transformation computation of boundary-layer equations between similarity regimes, *J. Comp. Phys.* 40 (1981) 478–490.
5. H.K. Kuiken, On boundary layers in fluid mechanics that decay algebraically along stretches of wall that are not vanishingly small, *IMA Jl. Appl. Math.* 27 (1981) 387–405.
6. D.A. Saville and S.W. Churchill, Laminar free convection in boundary layers near horizontal cylinders and vertical axisymmetric bodies, *J. Fluid Mech.* 29 (1967) 391–399.
7. T. Mahmood and J.H. Merkin, Mixed convection on a vertical cylinder, *J. Applied Math. and Physics (ZAMP)* 39 (1988) 186–203.
8. J.H. Merkin and T. Mahmood, Mixed convection boundary-layer similarity solutions: prescribed wall heat flux, *J. Applied Math. and Physics (ZAMP)* 40 (1989) 51–68.
9. E.T. Copson, *An Introduction to the theory of functions of a complex variable*, Clarendon Press, Oxford (1962).
10. K. Stewartson, *The theory of laminar boundary layers in compressible fluids*, Oxford Mathematical Monograph, O.U.P. (1964).
11. K. Stewartson, On asymptotic expansions in the theory of boundary layers, *J. Math. Phys.* 36 (1957) 173–191.
12. E.M. Sparrow, S.V. Patankar and R.M. Abdel-Wahed, Development of wall and free plumes above a heated vertical plate, *Trans. ASME, Journal of Heat Transfer* 100 (1978) 184–190.